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ALGORITHMS ON CLIQUE SEPARABLE GRAPHS

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We define a family of graphs, called the *clique separable graphs*, characterized by the fact that they have completely connected cut sets by which we decompose them into parts such that when no further decomposition is possible we have a set of simple subgraphs. For example the chordal graphs and the i-triangulated graphs are clique separable graphs.

The purpose of this paper is to describe polynomial time algorithms for the recognition of the clique separable graphs and for finding them a minimum coloring and a maximum clique.

1. Introduction

In this paper, we consider only finite, undirected graphs $G(V)$ with no parallel edges and no self-loops, where V is the set of the graph vertices. Two vertices of G connected by an edge are called *adjacent vertices*. For a subset U of V, the *subgraph* $G(U)$ of $G(V)$ is the graph whose set of vertices is U, two vertices being adjacent in $G(U)$ if and only if they are adjacent in $G(V)$. The *complement* G' of a graph $G(V)$ is the graph having the same set of vertices as $G(V)$, two vertices being adjacent in G' if and only if they are not adjacent in $G(V)$.

For two sets X, Y we denote by $X - Y$ the set of elements of X which are not in Y. For a set X we denote by $|X|$ the number of its elements.

For a connected graph $G(V)$, a subset A of V is called a *cut-set* of $G(V)$ if the subgraph $G(V - A)$ is disconnected. Consider a connected graph $G(V)$ having a completely connected cut-set A, and let $V_1, \ldots, V_s$ be the vertex sets of the connected components of $G(V - A)$. The subgraphs $G(V_1 \cup A), \ldots, G(V_s \cup A)$ are called the *leaves* of G produced by A.

A *completely connected set* of a graph $G(V)$ is a set of vertices whose every two elements are adjacent. A *clique* is a maximal completely connected set of vertices; a *maximum clique* is one with a maximum number of elements of all the cliques. A set of vertices of $G(V)$ is called *independent* if no two of its elements are adjacent. A *coloring* of $G(V)$ is obtained by assigning a color to every vertex such that no

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two adjacent vertices have the same color; a *minimum coloring* is one using the minimum number of colors of all the colorings.

In a directed tree T, if there is a directed path from a vertex u to a vertex v, then u is called an *ancestor* of v, and v is called a *descendant* of u. If u and v are connected by an edge directed from u to v, then u is called the *father* of v and v is called a *son* of u. If a vertex of T has no sons it is called *terminal*, otherwise it is called *non-terminal*.

A graph is called *bipartite* if its set of vertices can be partitioned in two disjoint subsets V_1, V_2 such that every edge of the graph connects between a vertex of V_1 and a vertex of V_2; i.e., V_1 and V_2 are independent sets.

Consider a simple circuit of a graph. A *chord* is an edge connecting two non-consecutive vertices of the circuit.

A graph is called *i-triangulated* if every odd simple circuit with more than three vertices has a set of chords which form with the circuit a planar graph such that the unbounded face is the exterior of the circuit, and all the bounded faces are triangles. These graphs were first discussed by Gallai [3] and Suranyi [7]. A short review can be found in Berge [1]. It is easy to prove (see Berge [1]) that a graph is i-triangulated if and only if every simple circuit of odd length t has $t-3$ chords that do not cross one another.

A graph is called *chordal* if every simple circuit with more than three vertices has a chord. These graphs were first discussed by Dirac [2]. Rose [6] described a recognition algorithm of these graphs. In [4] are described algorithms for finding a minimum coloring and a maximum clique of the chordal graphs and their complements.

A graph $G(V)$ is said to be of *Type* 1 if V can be partitioned in two disjoint subsets V_1, V_2 such that $|V_1| \geq 3$, $G(V_1)$ is a connected bipartite graph, V_2 is completely connected and every vertex of V_1 is adjacent to every vertex of V_2.

Clearly, a graph $G(V)$ is of Type 1 if and only if its complement G' is composed by a set of vertices with no incident edges and a subgraph which is the complement of a connected bipartite graph with at least three vertices. Therefore, we can easily recognize a graph of Type 1.

A graph $G(V)$ is said to be of *Type 2* if its set of vertices can be partitioned into disjoint subsets $V_1, \ldots, V_k$, every V_i being an independent set, such that for every $i \neq j$, every vertex of V_i is adjacent to every vertex of V_j. In fact a graph is of Type 2 if it is a complete k-partite graph for some k. It is easy to see that a graph is of Type 2 if and only if all the connected components of its complement are cliques.

We will define now a family of graphs called *clique separable graphs* by the following inductive definition:

a) A graph of Type 1 or 2 is a clique separable graph.

b) If a graph $G(V)$ has a completely connected cut-set A and the leaves of $G(V)$ produced by A are clique separable graphs, then $G(V)$ is a clique separable graph.

Clearly, every clique separable graph is connected. For example, the graph $G(V)$

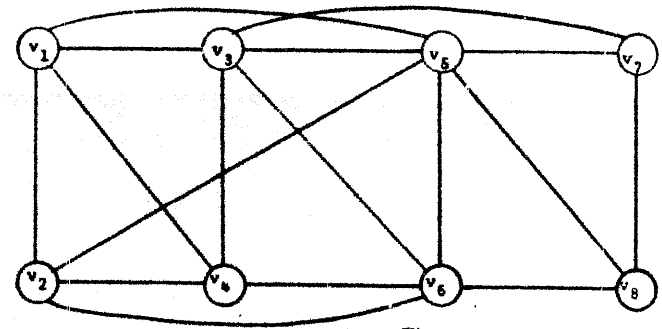

Fig. 1.

of Fig. 1 is a clique separable graph since it has the completely connected cut-set $\{v_3, v_5, v_6\}$, the leave $G(\{v_3, v_5, v_6, v_7, v_8\})$ is of Type 1, and the leave $G(\{v_1, v_2, v_3, v_4, v_5, v_6\})$ is of Type 2. These graphs appeared first in Gallai [3], without name. Gallai proved that every i-triangulated graph is a clique separable graph. The converse is not true since for example in the graph of Fig. 1 the circuit $v_3, v_4, v_6, v_8, v_7, v_3$ has only one chord, hence the graph is not i-triangulated. The clique separable graphs are perfect in the sense of Berge (see Berge [1]). Since Rose [6] proved that every chordal graph which is not completely connected has a completely connected cut-set, it follows that every chordal graph is a clique separable graph. By the definition, the bipartite graphs are also clique separable graphs. Therefore, the family of the clique separable graphs is quite large.

The purpose of this paper is to describe a polynomial time recognition algorithm for the clique separable graphs. Based on this, it will be easy to describe efficient algorithms for finding a minimum coloring and a maximum clique of such graphs. By my best knowledge, no such algorithms were previously known, since it is not easy to find out if a graph has a completely connected cut-set.

2. A recognition algorithm

Lemma. *Let A be a completely connected cut-set of a connected graph $G(V)$ and let D be a cut-set of a leave produced by A. Then D is a cut-set of $G(V)$.*

Proof. Let us assume that D is a cut-set of a leave $G(V_i \cup A)$, and let us denote by $U_1, \ldots, U_r$ the vertex sets of the connected components of $G(V_i \cup A - D)$. Clearly, D is different from A, otherwise it would not be a cut-set of $G(V_i \cup A)$.

Since A is completely connected, there is exactly one $j, 1 \leq j \leq r$, such that U_j contains the set $A - D$. Therefore, every path in $G(V)$ from a vertex of $A - D$ to a vertex of U_t, $t \neq j$, must contain a vertex of D. Hence D is a cut-set of $G(V)$.

Consider a connected graph $G(V)$ and let us construct a directed tree T_G as follows: The root of T_G is denoted by $G(V)$. If G has a completely connected cut-set, then set the leaves produced by the cut-set as sons of the vertex denoted $G(V)$ in T_G. Continue in this way on the descendants until no terminal vertex of T_G

has a completely connected cut-set. To every non-terminal vertex of T_G corresponds at least one pair of non adjacent vertices of $G(V)$, and to different nonterminal vertices of T_G correspond different pairs of non adjacent vertices of $G(V)$. Since the number of pairs of non adjacent vertices of $G(V)$ is at most $|V|^2/2$ it follows that T_G has at most $|V|^2$ vertices.

Theorem. *Consider a clique separable graph $G(V)$ which is not of Type 1 or 2. Then every clique C of $G(V)$ satisfies at least one of the following conditions:*

(a) *C is a cut-set of $G(V)$.*

(b) *There are two vertices $u, v \in C$ and two adjacent vertices $w, s \notin C$, such that $(C - \{u, v\}) \cup \{w, s\}$ is a completely connected cut-set of $G(V)$.*

(c) *There are two vertices $u \in C$, $v \notin C$ such that $(C - \{u\}) \cup \{v\}$ is a completely connected cut-set of $G(V)$.*

(d) *The subgraph defined by $\bar{C} = C \cup \{v \mid v \in V$, v is adjacent to $|C| - 1$ vertices of $C\}$ is of Type 2 and there is a subset $X \subset \bar{C}$ such that X is a completely connected cut-set of $G(V)$ and X is not a cut-set of $G(\bar{C})$.*

Proof. Assume that C is not a cut-set of $G(V)$ and construct T_G. Then, there is a subgraph $G(U)$ denoting a terminal vertex of T_G which contains C.

Assume that $G(U)$ is of Type 1 and let U_1, U_2 be the disjoint partition of U such that $G(U_1)$ is connected bipartite, $|U_1| \geq 3$, U_2 is completely connected and every vertex of U_1 is adjacent to every vertex of U_2. Since C is a clique, there are two vertices $u, v \in U_1$ such that $C = U_2 \cup \{u, v\}$. Clearly $|C| < |U_1| + |U_2|$ since $|U_1| \geq 3$. Let $G(\bar{U})$ be the father of $G(U)$ in T_G and let A be the completely connected cut-set of $G(\bar{U})$ used for constructing its sons. Then, there are two vertices $w, s \in U_1$ such that $A \subseteq U_2 \cup \{w, s\}$. Therefore $(C - \{u, v\}) \cup \{w, s\}$ is a cut-set of $G(\bar{U})$, since it cuts between the vertices of $U_1 - \{w, s\}$ and of $\bar{U} - U$. Hence by the Lemma it is a cut-set of $G(V)$, and C fulfills the condition (b).

Assume that $G(U)$ is of Type 2, and let $U_1, \ldots, U_k$ be the disjoint partition of U as required by Type 2.

Clearly, for every $1 \leq i \leq k$, $|U_i \cap C| = 1$ and every vertex of $U - C$ is adjacent to exactly $|C| - 1$ vertices of C. Let us assume that there exists a vertex of $V - U$ adjacent to $|C| - 1$ vertices of C. This vertex must be split from U in some ancestor of $G(U)$ in T_G. Let $G(\bar{U})$ be the closest ancestor of $G(U)$ whose sons split between U and a vertex $v \in V - U$ adjacent to $|C| - 1$ vertices of C. Let B be the completely connected cut-set of $G(\bar{U})$ which performs the splitting.

Since v is connected to $|C| - 1$ vertices of C it follows that $|C \cap B| \geq |C| - 1$. If $B \subseteq U$ then clearly $B = C$ or $|B - C| = 1$ and hence C fulfills the condition (a) or (c). Otherwise, let $u \in B - U$. Clearly u is connected to $|C| - 1$ vertices of C and u is a vertex of the son of $G(\bar{U})$ which son is also an ancestor of $G(U)$. Hence u must be split from U in a vertex of T_G closer than $G(\bar{U})$ to $G(U)$, contradicting our assumption on $G(\bar{U})$.

Therefore U is exactly $C \cup \{v \mid v \in V$, v is adjacent in $G(V)$ to $|C| - 1$ vertices of $C\}$. The completely connected cut-set used by the father of $G(U)$ for splitting, is also a cut-set of $G(V)$, it is different from C and it is not a cut-set of $G(U)$. Therefore C fulfills the condition (d).

Let us describe a polynomial time recognition algorithm for the clique separable graphs, using the above Theorem. Consider a connected graph $G(V)$. During the algorithm we construct the tree T_G. Denote the root of T_G by $G(V)$. If $G(V)$ is of Type 1 or 2 stop. Otherwise, construct a clique C of $G(V)$. If C fulfills any of the conditions (a), (b), (c), we easily find a completely connected cut-set of $G(V)$, and we add the produced leaves as the sons of $G(V)$ in T_G. Let us assume that C does not fulfill any of the conditions (a), (b), (c). Let

$$\bar{C} = C \cup \{v \mid v \in V,\ v \text{ is adjacent to } |C| - 1 \text{ vertices of } C\}.$$

If $G(\bar{C})$ is not of Type 2, then stop, G is not a clique separable graph. Assume that $G(\bar{C})$ is of Type 2. We have two cases:

Case 1. $G(V-\bar{C})$ is connected. Let $L = \{v \mid v \in \bar{C},\ v \text{ is adjacent to a vertex of } V-\bar{C}\}$. Since $G(V-\bar{C})$ is connected, any cut-set X of $G(V)$ such that $X \subset \bar{C}$ and X is not a cut-set of $G(\bar{C})$ must contain L. Therefore, if $L \neq \bar{C}$ and L is a completely connected cut-set of $G(V)$, then construct the leaves and add them as sons of $G(V)$ in T_G. Otherwise stop, $G(V)$ is not a clique separable graph.

Case 2. $G(V-\bar{C})$ is disconnected and let $V_1, \ldots, V_r$, $r \geq 2$, be the vertex sets of the connected components of $G(V-\bar{C})$. For every $1 \leq i \leq r$ let $X_i = \{v \mid v \in \bar{C},\ v$ is adjacent to a vertex of $V_i\}$. If there is an i such that X_i is a completely connected cut-set of $G(V)$, then construct the leaves and add them as sons of $G(V)$ in T_G. Otherwise stop, $G(V)$ is not a clique separable graph.

We continue in the same way on the descendants of $G(V)$ until all the terminal vertices of T_G are of Type 1 or 2, or until we stop because of $G(V)$ not being a clique separable graph.

The above algorithm can be implemented in $O(|V|^5)$.

Consider for example the graph $G(V)$ of Fig. 1. The set $C = \{v_1, v_2, v_4\}$ is a clique of $G(V)$ and it does not fulfil any of the conditions (a), (b), (c) of the Theorem. The set of vertices of $G(V)$ adjacent to $|C|-1$ vertices of C is $\{v_3, v_5, v_6\}$ hence $\bar{C} = \{v_1, v_2, v_4, v_3, v_5, v_6\}$. The subgraph $G(\bar{C})$ is of Type 2 and $G(V-\bar{C})$ is connected. The set of vertices of $\bar{C}$ adjacent to vertices of $V-\bar{C}$ is $L = \{v_3, v_5, v_6\}$. We can see that L is a completely connected cut-set of $G(V)$. Therefore we start constructing T_G as follows. The root of T_G is denoted by $G(V)$, and we add two sons denoted by the two subgraphs $G(V_1)$, $G(V_2)$, where $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V_2 = \{v_3, v_5, v_6, v_7, v_8\}$. The subgraph $G(V_1)$ is of Type 2 and the subgraph $G(V_2)$ is of Type 1 Hence $G(V)$ is a clique separable graph.

3. Minimum coloring and maximum clique

Consider a clique separable graph $G(V)$. We construct the tree T_G as in the recognition algorithm. Based on T_G we can find a minimum coloring and a maximum clique of $G(V)$ as follows. Let $G(U)$ be a terminal vertex of T_G. If $G(U)$ is of Type 1 then U can be partitioned in two disjoint subsets U_1, U_2 such that $|U_1| \geq 3$, $G(U_1)$ is connected bipartite, U_2 is completely connected and every vertex of U_1 is adjacent to every vertex of U_2. In this case a minimum coloring of $G(U)$ can be obtained by using $|U_2|$ different colors to color the vertices of $|U_2|$ and two additional colors to color the vertices of U_1, totally $|U_2|+2$ colors. A maximum clique having $|U_2|+2$ elements can be obtained by adding to U_2 any two adjacent vertices of U_1.

If $G(U)$ is of Type 2 then it has a partition into a family of disjoint subsets $U_1, U_2, \ldots, U_k$ such that every U_i is independent and for $i \neq j$, every vertex of U_i is adjacent to every vertex of U_j. Therefore, we obtain a minimum coloring by using k colors and coloring the entire set U_i with the i-th color, for every $1 \leq i \leq k$. A maximum clique of k vertices is obtained by taking exactly one vertex from every U_i. In this way we obtain minimum colorings and maximum cliques for all the subgraphs corresponding to the terminal vertices of T_G. Now we go up towards the root of T_G as follows.

Consider a vertex $G(U)$ of T_G such that we have already a minimum coloring and a maximum clique for every son of $G(U)$ in T_G. Let $G(U_1), \ldots, G(U_r)$ be these sons, and let A be the completely connected cut-set which gave the splitting of $G(U)$ into these leaves. A maximum clique of $G(U)$ is obtained by taking a clique of maximum size among the maximum cliques of $G(U_1), \ldots, G(U_r)$. We obtain a minimum coloring of $G(U)$ as follows. Let us assume that for two different sons of $G(U)$ we used colors with different names. Consider the son $G(U_t)$ of $G(U)$ whose minimum coloring uses the maximum number of colors among the minimum colorings of all the sons of $G(U)$. We know that for every $1 \leq i \leq r$, $A \subset U_i$ and for every $j \neq i$, $(U_i - A) \cap (U_j - A) = \emptyset$. For every $j \neq t$ we rename the colors used by $G(U_j)$ on A by the names of the colors used by $G(U_t)$ on A. Also we rename the other colors of $G(U_j)$ by names of colors of $G(U_t)$ not used on A. Thus we obtain a minimum coloring of $G(U)$ using only the colors used by the minimum coloring of $G(U_t)$

In this way we continue until we obtain a minimum coloring and a maximum clique of $G(V)$. In fact we can obtain a maximum clique and the number of colors in a minimum coloring of $G(V)$ directly from the terminal vertices of T_G.

4. Conclusions

We described polynomial time algorithms for the recognition of the clique separable graphs and for finding minimum colorings and maximum cliques. Yet, many related questions remain unanswered. A polynomial time algorithm for finding a minimum covering by cliques and a maximum independent set of a clique separable graph is unknown. Another problem is to characterize these graphs similarly to the i-triangulated graphs. The secret may be hid in the paper of Gallai [3]. I can make only the following conjecture:

Conjecture. *A graph $G(V)$ is a clique separable graph if and only if every odd simple circuit has a set of chords which forms with the circuit a planar graph such that the unbounded face is the exterior of the circuit and all the bounded faces have boundaries which are triangles or simple circuits of even length.*

An interesting problem is also to describe an efficient algorithm for the recognition of the i-triangulated graphs.

Let us now enlarge the family of clique separable graphs by assuming as primitive not only the graphs of Type 1 or 2, but all the transitive orientable graphs (for definition see [5]). The problem is to find a polynomial time algorithm for the recognition of these graphs.

In general, we can ask how do we characterize the perfect graphs which have no completely connected cut-sets. It seems that this question is related to the conjecture that a graph is perfect if and only if nor it nor its complement has a simple circuit without chords as a subgraph.

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